Motivation towards Clifford Analysis: The classical Cauchy's Integral Theorem from a Clifford perspective

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Abstract

Clifford Algebras generalise complex variables algebraically and analytically. In particular, this includes a generalisation of the notion of holomorphy and Cauchy's Integral Theorem. We give a brief overview of the general theory, before proving the classical Cauchy's Integral Theorem via differential forms and Stokes' Theorem, to give a Clifford approach to complex variables that lends itself to generalisation.

1 An overview of Clifford Analysis

We let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. We construct the *Clifford Algebra* over $\mathbb{F}$ as follows. Let $\{e_0, e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{R} \times \mathbb{R}^n$. We define multiplication of these basis elements in the following way:

$$\begin{aligned} e_0 &= 1 & & \\ e_j^2 &= -e_0 & & 1 \leq j \leq n \\ e_i e_j &= -e_j e_i & & 1 \leq i < j \leq n \end{aligned}$$

Whenever $1 \leq j_1 < j_2 < \cdots < j_k \leq n$, write $S = \{j_1, j_2, \ldots, j_k\}$ and define $e_S = e_{j_1} e_{j_2} \ldots e_{j_k}$. For $S = \varnothing$, define $e_\varnothing = e_0$. The Clifford algebra $\mathbb{F}_{(n)}$ is defined as

$$\mathbb{F}_{(n)} = \mathbb{F}\text{-}\operatorname{span}\{e_S : S \subset \{1, \ldots, n\}\}$$

which makes it a 2^n-dimensional algebra. If $u, v \in \mathbb{F}_{(n)}$, then $u = \sum_S u_S e_S$ and $v = \sum_R u_R e_R$ with $u_S, v_R \in \mathbb{F}$ and $uv = \sum_{S,R} u_S v_R e_S e_R$. The term u_0 is the *scalar* part of u.

We equip the algebra with an involution. The *Clifford conjugate* for a basis element e_S is the element $\overline{e}_S$ satisfying $\overline{e}_S e_S = 1 = e_0 = e_S \overline{e}_S$. Observe then that $\overline{e}_S = \pm e_S$, with the sign chosen appropriately. Then, the Clifford conjugate of a general $u \in \mathbb{F}_{(n)}$ is defined as $\overline{u} = \sum_S \overline{u}_S \overline{e}_S$. A calculation reveals that $\overline{uv} = \overline{v}\,\overline{u}$. Also, note that $u\overline{v} = \sum_S u_S \overline{v}_S + \sum_{S \neq R} u_S \overline{v}_R e_S \overline{e}_R$. From this observation we define an inner product $\langle u, v \rangle = (u\overline{v})_0 = \sum_S u_S \overline{v}_S$. We write $|\cdot| = |\cdot|_2$ for the associated norm.

We can embed $\mathbb{R}^{n+1}$ in $\mathbb{F}_{(n)}$ by identifying it with the subspace $\mathbb{R}\text{-}\operatorname{span}\{e_0, \ldots, e_n\}$ via the map $(x_0, \ldots, x_n) \mapsto \sum_{j=0}^n x_j e_j$. Then, whenever $m \leq n$, we can consider $\mathbb{R}^m \subset \mathbb{R}^{n+1}$ by identifying $\mathbb{R}^m$ with the subspace $\operatorname{span}\{e_1, \ldots, e_m\}$ and so via transitivity, we can embed $\mathbb{R}^m$ in $\mathbb{F}_{(n)}$.

Not every element of $\mathbb{F}_{(n)}$ is invertible [MP87]. However, if $x \in \mathbb{R}^{n+1}$, then it does have an inverse. An easy calculation shows that $\overline{e}_j = -e_j$ whenever $1 \leq j \leq n$. Given an element $x \in \mathbb{R}^{n+1}$ the conjugate $\overline{x} = x_0 - \sum_{j=1}^{n} x_j e_j$. The *Kelvin inverse* of x is then given by

$$x^{-1} = \frac{\overline{x}}{|x|^2} = \frac{x_0 - \sum_{j=1}^{n} x_j e_j}{\sum_{j=0}^{n} x_j^2}.$$

We highlight and important fact. The Clifford algebra $\mathbb{R}_{(1)}$ can be identified with the Complex numbers. This can be seen by identifying $e_0 \mapsto 1$ and $\imath \mapsto e_1$. We emphasise that this is an algebra isomorphism over $\mathbb{R}$. It is straightforward to check that the Clifford conjugate agrees with the classical complex conjugate. Furthermore, every element in $\mathbb{R}_{(1)}$ is invertible, because in the case $n = 1$, there is a vector space isomorphism $\mathbb{R}_{(1)} \cong_{\mathbb{R}} \mathbb{R}^2$. We also note that the algebra $\mathbb{R}_{(2)}$ can be identified with the *Quaternions*.

A *Banach module* over $\mathbb{F}_{(n)}$ is a Banach space $\mathcal{X}$ over $\mathbb{F}$ with an operation of multiplication by elements of $\mathbb{F}_{(n)}$ with a $\kappa \geq 1$ such that

$$\|xu\| \leq \kappa |u| \|x\| \qquad \text{and} \qquad \|ux\| \leq \kappa |u| \|x\|$$

for all $x \in \mathcal{X}$ and $u \in \mathbb{F}_{(n)}$. In some sense, a Banach module is a generalisation of the concept of a space over a field, since in the module we are able to multiply by elements of $\mathbb{F}_{(n)}$.

Suppose $\mathcal{X}$ and $\mathcal{Y}$ are Banach modules over $\mathbb{F}_{(n)}$. We say that $A : \mathcal{X} \to \mathcal{Y}$ is a *right module homomorphism* if $(Ax)u = A(xu)$ for all $x \in \mathcal{X}$ and $u \in \mathbb{F}_{(n)}$. Certainly, we also have a notion of *left module homomorphism*. Namely, it is a map $B : \mathcal{X} \to \mathcal{Y}$ satisfying $(ux)B = u(xB)$. That this situation arises can be seen by considering the operator $A = \sum_{j=0}^{n} A_j e_j$, with $A_j \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then, in general, we do not expect $xA = Ax$. The space of continuous right homomorphisms is denoted $\mathcal{L}_{(n)}(\mathcal{X}, \mathcal{Y})$ and it is considered as a Banach space with the uniform operator topology.

Let $\mathcal{X}_{(n)} = \mathcal{X} \otimes \mathbb{F}_{(n)}$. Every $\xi \in \mathcal{X}_{(n)}$ arises in the form $\xi = \sum_S x_S \otimes e_S$, where $x_S \in \mathcal{X}$. For convenience, we omit the $\otimes$ and simply write $\xi = \sum_S x_S e_S$. Multiplication by $u \in \mathbb{F}_{(n)}$ is defined by $\xi u = \sum_{S,R} u_R x_S e_S e_R$ and $u\xi = \sum_{S,R} u_R x_S e_R e_S$. We equip $\xi \in \mathcal{X}_{(n)}$ with the norm $\|\xi\| = (\sum_S \|x_S\|^2)^{\frac{1}{2}}$. It follows then that $\mathcal{X}_{(n)}$ is a Banach module with $\kappa = 1$. We highlight a trivial fact. Since $\mathbb{F}$ is a Banach space over $\mathbb{F}$, the module $\mathbb{F} \otimes \mathbb{F}_{(n)} \cong \mathbb{F}_{(n)}$ is in fact a Banach module over $\mathbb{F}_{(n)}$. This observation shows our choice of notation for a Clifford algebra is consistent with that of a Banach module. Furthermore, the space $(\mathcal{L}(\mathcal{X}, \mathcal{Y}))_{(n)}$ can be identified with $\mathcal{L}_{(n)}(\mathcal{X}_{(n)}, \mathcal{Y}_{(n)})$ [Jef04, §3.2].

Now we can begin to consider some analytic properties of Clifford valued functions. Our approach is taken from [MP87] and [Jef04]. A more measure theoretic approach can be found in [Mit94, §1.2].

Given $\Omega \subset \mathbb{R}^{n+1}$ an open set, any function $f : \Omega \to \mathbb{F}_{(n)}$ can be written as $f = \sum_S f_S e_S$. Often, we regard f as a function $f : \Omega \subset \mathbb{F}_{(n)} \to \mathbb{F}_{(n)}$ via the canonical embedding. We say $f \in C^\infty(\Omega, \mathbb{F}_{(n)})$ if $f_S \in C^\infty(\Omega)$ for every S. For such a function, letting ∂_j denote the partial derivative in the direction j, we define

$$D = \partial_0 e_0 + \sum_{j=1}^{n} \partial_j e_j.$$

Then, $Df = \sum_S (\partial_0 f_S e_S + \sum_{j=1}^{n} \partial_j f_S e_j e_S)$ and $fD = \sum_S (\partial_0 f_S e_S + \sum_{j=1}^{n} \partial_j f_S e_S e_j)$. Such a function is said to be *left monogenic* if $Df = 0$ and *right monogenic* if $fD = 0$ on Ω. In the case of $\mathbb{R}_{(1)} \cong_R \mathbb{C}$, we find that $Df = fD$ and $Df = 0$ are exactly the holomorphic functions. The notion of monogenic generalises holomorphy.

Let $(\Sigma, \mathscr{M}, \mu)$ be a measure space. Then, the integral of a μ-measurable function f over a set Ω is given by

$$\int_\Omega f \ d\mu = \sum_S (\int_\Omega f_S \ d\mu) e_S.$$

With this in mind, we have the following result as a first step towards a generalisation of Cauchy's Integral Theorem [FB82, §8.8].

Theorem 1.1 (Existence and uniqueness of fundamental solution)**.** *There exists a unique left and right fundamental solution $E \in L^1_{loc}(\mathbb{R}^{n+1}, \mathbb{F}_{(n)})$ in the sense of distributions $\mathscr{S}(\mathbb{R}^{n+1})_{(n)}$ to the equation $DE = ED = \delta_0 e_0$. When $x \neq 0$, the fundamental solution is given by*

$$E(x) = \frac{1}{\omega_{n+1}} \frac{\overline{x}}{|x|^{n+1}}$$

where $\frac{1}{\omega_{n+1}} = \frac{1}{2}\pi^{-\frac{n+1}{2}}\Gamma(\frac{n+1}{2})$, the area of the unit sphere S^n. Furthermore, E is both left and right monogenic on $\mathbb{R}^{n+1} \setminus \{0\}$.

A detailed treatment of distributions taking values in Banach modules is found in [FB82, §2]. From the fundamental solution, we can construct a generalised *Cauchy kernel.* Analogous to the case of complex variables, we write the Cauchy kernel as $G_x(\omega) = E(\omega - x)$ for all $\omega \neq x$ and the following theorem shows that it is indeed a generalisation of the classical Cauchy kernel.

Theorem 1.2 (Monogenic Cauchy's integral theorem)**.** *Suppose that $\Omega \subset \mathbb{R}^{n+1} \subset \mathbb{F}_{(n)}$ is a bounded open set with a smooth boundary $\partial\Omega$, and $\nu : \mathbb{R}^{n+1} \subset \mathbb{F}_{(n)} \to \mathbb{R}^{n+1} \subset \mathbb{F}_{(n)}$ is the unit outer normal to $\partial\Omega$. Furthermore, let μ denote the surface measure on $\partial\Omega$. Suppose $f, g : \Omega' \to \mathbb{F}_{(n)}$ where $\overline{\Omega} \subset \Omega'$ and Ω' are open sets. If f is left monogenic, and g is right monogenic. Then the following hold:*

1. *$\int_{\partial\Omega} G_x(\omega)\nu(\omega)f(\omega) \ d\mu(\omega) = f(x)$ when $x \in \Omega$ and 0 otherwise,*
2. *$\int_{\partial\Omega} g(\omega)\nu(\omega)G_x(\omega) \ d\mu(\omega) = g(x)$ when $x \in \Omega$ and 0 otherwise,*
3. *$\int_{\partial\Omega} g(\omega)\nu(\omega)f(\omega) \ d\mu(\omega) = 0.$*

2 The case of $\mathbb{R}_{(1)}$

We now focus our attention on $\mathbb{R}_{(1)}$ and prove Theorem 1.2 for when $n = 1$ using differential forms to be adequately general. For classical reasons, the directions e_0 and e_1 are respectively associated with the variables x and y. Then,

$$D = \partial_x e_0 + \partial_y e_1.$$

We emphasise that we always regard $\mathbb{R}^2 \hookrightarrow \mathbb{R}_{(1)}$.

We firstly show that we have a product rule for D.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^2$ be an open set and let $f, g : \Omega \to \mathbb{R}_{(1)}$ be differentiable. Then $D(fg) = (Df)g + f(Dg)$.*

Proof. Firstly, we write $f = f_0 e_0 + f_1 e_1$ and $g = g_0 e_0 + f_1 e_1$. Then,

$$Df = (\partial_x e_0 + \partial_y e_1)(f_0 e_0 + f_1 e_1) = (\partial_x f_0 - \partial_y f_1)e_0 + (\partial_x f_1 + \partial_y f_0)e_1$$

and

$$fg = (f_0g_0 - f_1g_1)e_0 + (f_1g_0 + f_0g_1)e_1.$$

We compute $D(fg)$:

$$\begin{aligned}D(fg) &= [\partial_x(f_0g_0 - f_1g_1) - \partial_y(f_1g_0 + f_0g_1)]e_0 + [\partial_x(f_1g_0 + f_0g_1) + \partial_y(f_0g_0 - f_1g_1)]e_1\\ &= [\partial_x f_0 g_0 + f_0\partial_x g_0 - \partial_x f_1 g_1 - f_1\partial_x g_1 - \partial_y f_1 g_0 - f_1\partial_y g_0 - \partial_y f_0 g_1 - f_0\partial_y g_1]e_0 +\\ &\qquad [\partial_x f_1 g_0 + f_1\partial_x g_0 + \partial_x f_0 g_1 + f_0\partial_x g_1 + \partial_y f_0 g_0 + f_0\partial_y g_0 - \partial_y f_1 g_1 - f_1\partial_y g_1]e_1\end{aligned}$$

Also,

$$(Df)g = [(\partial_x f_0 - \partial_y f_1)g_0 - (\partial_x f_1 + \partial_y f_0)g_1]e_0 + [(\partial_x f_1 + \partial_y f_0)g_0 + (\partial_x f_0 - \partial_y f_1)g_1]e_1$$

and by interchanging f and g,

$$(Dg)f = [(\partial_x g_0 - \partial_y g_1)f_0 - (\partial_x g_1 + \partial_y g_0)f_1]e_0 + [(\partial_x g_1 + \partial_y g_0)f_0 + (\partial_x g_0 - \partial_y g_1)f_1]e_1$$

It is then a simple but tedious task to compare the expressions to conclude $D(fg) = (Df)g + f(Dg)$. $\square$

As a consequence of Theorem 1.1, the Cauchy kernel $G_p(\omega)$ is simply a translation of of E and we have the following Corollary.

Corollary 2.2. *Let $f = G_p$ and suppose g is left monogenic. Then, $D(G_pg) = (DG_p)g$ on $\Omega\setminus\{p\}$.*

The following is a Stokes' type theorem for the operator D.

Proposition 2.3. *Let $\Omega, \Omega' \subset \mathbb{R}^2$ be open sets such that $\overline{\Omega} \subset \Omega'$ and Ω is bounded with smooth boundary $\partial\Omega$. Suppose also that $\partial\Omega$ is equipped with unit outer normal ν and surface measure μ. Let $f \in C^1(\Omega', \mathbb{R}_{(1)})$. Then,*

$$\int_\Omega D(G_pf)\ d\mathscr{L} = \int_{\partial\Omega} G_p\nu f\ d\mu$$

for all $p \in \Omega$.

Proof. First, let $\theta_0 = (G_pf)_0 = (G_p)_0f_0 - (G_p)_1f_1$ and $\theta_1 = (G_pf)_1 = (G_p)_0f_1 + (G_p)_1f_0$. It then follows that $D(G_pf) = (\partial_x\theta_0 - \partial_y\theta_1)e_0 + (\partial_x\theta_1 + \partial_y\theta_0)e_1$ and therefore,

$$\int_\Omega D(G_pf)\ d\mathscr{L} = \left[\int_\Omega (\partial_x\theta_0 - \partial_y\theta_1)\ d\mathscr{L}\right]e_0 + \left[\int_\Omega (\partial_x\theta_1 + \partial_y\theta_0)\ d\mathscr{L}\right]e_1.$$

Define $\xi_1, \xi_2 \in \wedge^{n-1}(\partial\Omega)$ by $\xi_1 = \theta_1\ dx + \theta_0\ dy$ and $\xi_2 = -\theta_0\ dx + \theta_1\ dy$. Therefore, $d\xi_1 = (\partial_x\theta_0 - \partial_y\theta_1)\ dx\wedge dy$ and $d\xi_2 = (\partial_x\theta_1 + \partial_y\theta_0)\ dx\wedge dy$. By the definition of integration of an n-form and by the application of Stokes' Theorem,

$$\int_\Omega (\partial_x\theta_0 - \partial_y\theta_1)\ d\mathscr{L} = \int_\Omega (\partial_x\theta_0 - \partial_y\theta_1)\ dx\wedge dy = \int_\Omega d\xi_1 = \int_{\partial\Omega}\xi_1 = \int_{\partial\Omega}(\theta_1 dx + \theta_0 dy)$$

and by similar calculation

$$\int_\Omega (\partial_x\theta_1 + \partial_y\theta_0)\ d\mathscr{L} = \int_{\partial\Omega}(-\theta_0\ dx + \theta_1\ dy)$$

Let $\nu(p) = \nu_0(p)e_0 + \nu_1(p)e_1$. Then the orthogonal projection of ν given by $\nu^{\perp}(p) = -\nu_1(p)e_0 + \nu_0(p)e_1$ and $\nu^{\perp} \in \Gamma(\mathrm{T}(\partial\Omega))$. Letting $d\mu$ be the volume form on $\partial\Omega$, $d\mu(\nu^{\perp}) = 1$. Note that $\{-\nu_1(p)e_0, \nu_0(p)e_1\}$ form a basis for $\mathrm{T}_p(\partial\Omega)$ and it follows that $dx = -\nu_1 d\mu$ and $dy = \nu_0 d\mu$. It then follows that,

$$\begin{aligned}
\int_{\Omega} D(G_p f)\, d\mathscr{L} &= \left[\int_{\partial\Omega} (-\theta_1\nu_1 + \theta_0\nu_0)\, d\mu\right] e_0 + \left[\int_{\partial\Omega} (\theta_0\nu_1 + \theta_1\nu_0) d\mu\right] e_1 \\
&= \int_{\partial\Omega} (-\theta_1\nu_1 + \theta_0\nu_0)e_0 + (\theta_0\nu_1 + \theta_1\nu_0)e_1\, d\mu \\
&= \int_{\partial\Omega} (-(G_p)_0 f_1\nu_1 - (G_p)_1 f_0\nu_1 + (G_p)_0 f_0\nu_0 - (G_p)_1 f_1\nu_0)e_0 \\
&\qquad + ((G_p)_0 f_0\nu_1 - (G_p)_1 f_1\nu_1 + (G_p)_0 f_1\nu_0 + (G_p)_1 f_0\nu_0)e_1\, d\mu
\end{aligned}$$

and

$$\begin{aligned}
G_p\nu f = {} & [((G_p)_0\nu_0 - (G_p)_1\nu_1)f_0 - ((G_p)_1\nu_0 + (G_p)_0\nu_1)f_1]e_0 \\
& + [((G_p)_1\nu_0 + (G_p)_0\nu_1)f_0 + ((G_p)_0\nu_0 - (G_p)_1\nu_1)f_1]e_1.
\end{aligned}$$

The conclusion then follows by comparing these two calculations and since we can interchange between the surface measure and the surface (volume) form. □

Combining these results, we can prove the following version of Cauchy's integral theorem.

Theorem 2.4. *Let $\Omega, \Omega' \subset \mathbb{R}^2$ be open sets such that $\overline{\Omega} \subset \Omega'$ and Ω is bounded with smooth boundary $\partial\Omega$. Suppose also that $\partial\Omega$ is equipped with unit outer normal ν and surface measure μ. Let $f : \Omega' \to \mathbb{R}_{(1)}$ be left monogenic. Then,*

$$f(p) = \int_{\partial\Omega} G_p \nu f\, d\mu$$

for all $p \in \Omega$.

Proof. Since Ω and Ω' are open and $\overline{\Omega} \subset \Omega'$ means that there exists an $\varepsilon > 0$ such that $\overline{\Omega + \varepsilon} \subset \Omega'$. Let $\Omega_1 = \Omega + \frac{1}{2}\varepsilon$ and $\Omega_2 = \Omega + \varepsilon$. Then, there exists a function $\tilde{f} \in \mathscr{S}(\mathbb{R}^2)_{(1)}$ such that $\tilde{f} = f$ on Ω_1 and $\tilde{f} = 0$ on $\mathbb{R}^2 \setminus \Omega_2$.

Next, we observe that on Ω_1, $\tilde{f}$ is left monogenic on $\Omega_1 \setminus \{p\}$ we apply Corollary 2.2 combined with the fact that $\tilde{f} \in \mathscr{S}(\mathbb{R}^2)_{(1)}$ and $f = \tilde{f}$ on Ω_1,

$$\int_{\Omega} D(G_p f)\, d\mathscr{L} = \int_{\Omega} D(G_p \tilde{f})\, d\mathscr{L} = \int_{\Omega} (DG_p)\tilde{f} = \int_{\Omega} \tilde{f}\, d\delta_p = \tilde{f}(p) = f(p)$$

Then by application of Proposition 2.3,

$$f(p) = \int_{\Omega} D(G_p f)\, d\mathscr{L} = \int_{\partial\Omega} G_p \nu f\, d\mu$$

which concludes the proof of the theorem. □

We remark that the key elements of the proof were Proposition 2.1 and 2.3. Indeed, if we were able to generalise these two key results, the proof of the main theorem would be a proof for case (1) of Theorem 1.2 without alteration.

In the proof of Proposition 2.3, we took the smooth unit outer normal ν and then rotated it counter-clockwise by $\pi/2$ to obtain the perpendicular $\nu^{\perp} \in \Gamma(\mathrm{T}(\partial\Omega))$. In this case, we only have a single direction (up to orientation) to choose from. To extract out such a $\nu^{\perp}$ in a general setting, we would need to proceed by projecting ν to the directions at each p. Such would be a basis for $\mathrm{T}_p(\partial\Omega)$, and we would need to combine these in a way to obtain $d\mu(\nu^{\perp}) = 1$.

Also, notice that the Clifford multiplication captures the rotation of ν to obtain the corresponding tangential vector. Thus, we expect this should give us insight into how to construct the correct differential form in the general case.

We have yet to show the relationship between monogeniety and holomorphy, and also between the classical Cauchy Integral Theorem in Complex variables. Our discussion from this point onwards will be specific to $\mathbb{R}_{(1)}$.

Proposition 2.5. *f is a left monogenic function if and only if f is right monogenic.*

Proof. A simple calculation shows that $Df = (\partial_x f_0 - \partial_y f_1)e_0 + (\partial_x f_1 + \partial_y f_0)e_1 = fD$. $\square$

This justifies us calling a function monogenic rather than left/right monogenic. In light of this observation, the integral in Theorem 2.4 also holds for the right monogenic case. This is actually due to the fact that multiplication in $\mathbb{R}_{(1)}$ is commutative. Had we used this fact, the proof of Proposition 2.3 would have been simplified greatly. However, this would have been at the cost of losing scope of the general perspective.

We have the following important observation which illustrates the fact that monogeniety is holomorphy in a different (but isomorphic) algebraic setting.

Proposition 2.6. *A function f is monogenic if and only if f_0 and f_1 satisfy the Cauchy-Riemann equations.*

Proof. The proof of this is remarkably easy. Notice that $0 = Df = (\partial_x f_0 - \partial_y f_1)e_0 + (\partial_x f_1 + \partial_y f_0)e_1$ if and only if $0 = \partial_x f_0 - \partial_y f_1$ and $0 = \partial_x f_1 + \partial_y f_0$ which are exactly the Cauchy-Riemann equations. $\square$

Now, let $\mathcal{I} : \mathbb{R}_{(1)} \cong_{\mathbb{R}} \mathbb{C}$ denote the usual algebra isomorphism which is given by $\mathcal{I}(e_0) = 1$ and $\mathcal{I}(e_1) = \imath$. In light of this notation, a function f is monogenic if and only if $\mathcal{I} f \mathcal{I}^{-1}$ is holomorphic.

We require the following key change of coordinates formula for boundaries parametrised by smooth curves. Note that $\Gamma(\mathrm{T}(\partial\Omega))$ denotes sections over the bundle $\mathrm{T}(\partial\Omega)$.

Proposition 2.7. *Let $\Omega, \Omega' \subset \mathbb{R}^2$ be open sets such that $\overline{\Omega} \subset \Omega'$ and suppose that $\partial\Omega$ is smooth and that $\gamma : [0,1] \to \partial\Omega$ is is a unit speed parametrisation of $\partial\Omega$. As before, let ν be the unit outer normal and μ the surface measure on $\partial\Omega$. Then,*

$$\int_{\partial\Omega} \theta\nu \; d\mu = -e_1 \int_0^1 (\theta \circ \gamma) \; \gamma \; dt$$

for all $\theta \in C^1(\Omega', \mathbb{R}_{(1)})$.

Proof. First, we note that $\dot{\gamma} = \dot{\gamma}_0 e_0 + \dot{\gamma}_1 e_1 \in \Gamma(\mathrm{T}(\partial\Omega))$ and $|\dot{\gamma}| = 1$. The unit outer normal is then the rotation of $\dot{\gamma}$ by $-\pi/2$ and so it follows that $\nu \circ \gamma = \dot{\gamma}_0 e_0 - \dot{\gamma}_1 e_1$. With the observation that

$\nu \circ \gamma = -e_1 \dot{\gamma}$ and since γ is parametrised via arc length,

$$\int_{\partial\Omega} \theta\nu \; d\mu = \int_0^1 (\theta \circ \gamma) \; (\nu \circ \gamma) \; dt = \int_0^1 (\theta \circ \gamma) \; (-e_1 \dot{\gamma}) \; dt = -e_1 \int_0^1 (\theta \circ \gamma) \; \dot{\gamma} \; dt$$

which concludes the proof. $\square$

The following theorem then illustrates that Theorem 2.4 is really the classical Cauchy Integral Theorem in another language.

Proposition 2.8. *Let $\Omega, \Omega' \subset \mathbb{R}^2$ be open sets such that $\overline{\Omega} \subset \Omega'$ and suppose that $\partial\Omega$ is smooth and that $\gamma : [0,1] \to \partial\Omega$ is is a unit speed parametrisation of $\partial\Omega$. As before, let ν be the unit outer normal and μ the surface measure on $\partial\Omega$. Suppose $f : \Omega' \to \mathbb{R}_{(1)}$ is monogenic and let $\tilde{f} : \Omega' \hookrightarrow \mathbb{C} \to \mathbb{C}$ be given by $\tilde{f} = \mathcal{I} f \mathcal{I}^{-1}$. Then,*

$$(\tilde{f})(\xi) = \mathcal{I} \int_{\partial\Omega} G_{\mathcal{I}^{-1}(\xi)} f \nu \; d\mu = \frac{1}{2\pi i} \int_{\tilde{\gamma}} \frac{\tilde{f}(\zeta)}{\zeta - \xi} \; d\zeta$$

where $\tilde{\gamma} = \mathcal{I}\gamma$.

Proof. Fix $\xi \in \Omega$, let $p = \mathcal{I}^{-1}\xi$ and , let $\theta = G_p f = G_{\mathcal{I}^{-1}\xi} f$. Also, by Theorem 1.1,

$$G_p(x) = \frac{1}{2\pi} \frac{\overline{x - p}}{|x-p|^2} \qquad \Longleftrightarrow \qquad \mathcal{I} G_{\mathcal{I}^{-1}\xi} \mathcal{I}^{-1}(\zeta) = \frac{1}{2\pi} \frac{1}{\zeta - \xi}$$

and since multiplication in $\mathbb{R}_{(1)}$ is commutative, we apply Proposition 2.7 to find:

$$\begin{aligned} \mathcal{I} \int_{\partial\Omega} G_{\mathcal{I}^{-1}\xi} f \nu &= -\imath \int_0^1 \mathcal{I}(G_{\mathcal{I}^{-1}} f \circ \gamma) \; \mathcal{I}\dot{\gamma} \; dt \\ &= \frac{1}{\imath} \int_0^1 (\mathcal{I} G_{\mathcal{I}^{-1}} \mathcal{I}^{-1})(\mathcal{I}\gamma)(\mathcal{I} f \mathcal{I}^{-1})(\mathcal{I}\gamma) \; dt \\ &= \frac{1}{\imath} \int_0^1 (\mathcal{I} G_{\mathcal{I}^{-1}} \mathcal{I}^{-1})(\tilde{\gamma}) \tilde{f}(\tilde{\gamma}) \; dt \\ &= \frac{1}{2\pi\imath} \int \tilde{\gamma} \frac{\tilde{f}}{\zeta - \xi} \; d\zeta. \end{aligned}$$

$\square$

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