Rough metrics, the Kato square root problem, and the continuity of a flow tangent to the Ricci flow

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Motivation: the flow of Gigli-Mantegazza

Let $\mathcal{M}$ be a compact manifold with a smooth metric g. Let Δ_{g} be its Laplacian (on functions) and $\rho^{\mathrm{g}}_{\cdot}(\cdot, \cdot) \in \mathrm{C}^\infty(\mathbb{R}_+ \times \mathcal{M} \times \mathcal{M})$ denote the heat kernel.

Fix a point $x \in \mathcal{M}$ and a time $t > 0$, and two tangent vectors $u, v \in \mathrm{T}_x\mathcal{M}$. Let $\varphi_{t,x,v} \in \mathrm{L}^2(\mathcal{M})$ with $\int_{\mathcal{M}} \varphi_{t,x,v}\, d\mu_{\mathrm{g}} = 0$ be the solution to the PDE:

$$-\operatorname{div}_{\mathrm{g},y} \rho^{\mathrm{g}}_t(x,y) \nabla \varphi_{t,x,v}(y) = \mathrm{d}_x(\rho^{\mathrm{g}}_t(x,y))(v), \tag{GMC}$$

Gigli and Mantegazza in [GM] define $\mathrm{g}_t(x)$ on tangent vectors $u, v \in \mathrm{T}_x\mathcal{M}$ by the expression:

$$\mathrm{g}_t(x)(u,v) = \int_{\mathcal{M}} \mathrm{g}(y)(\nabla \varphi_{t,x,u}(y), \nabla \varphi_{t,x,v}(y))\ \rho^{\mathrm{g}}_t(x,y)\ d\mu_{\mathrm{g}}(y). \tag{GM}$$

This expression can indeed be checked to define an inner product on $\mathrm{T}_x\mathcal{M}$.

Moreover, Gigli and Mantegazza show that:

$$\partial_t \mathrm{g}_t(\dot\gamma(s),\dot\gamma(s))\big|_{t=0} = -2\,\mathrm{Ric}_{\mathrm{g}}(\dot\gamma(s),\dot\gamma(s)),$$

for almost-every s along g-geodesics γ. That is, g_t is tangential to the *Ricci flow* in this weak sense.

The defining equation (GMC) can be "lifted" to a distributional equation in *Wasserstein space*. Using the induced heat flow, we obtain a time evolving family of distance metrics d_t starting with the initial metric $\mathrm{d}_0 = \mathrm{d}_{\mathrm{g}}$, the induced distance from g. Indeed, d_t is induced from the metrics g_t defined by (GM).

In fact, the fact that (GMC) can be given meaning in Wasserstein space means exactly that the flow of distance metrics d_t can be defined for an RCD-space $(\mathcal{X}, \mathrm{d}, \mu)$ (a measure metric space with a notion of Ricci curvature bounded from below and with a Hilbertian Sobolev space).

In particular, this allows us to flow spaces containing singularities. Given that there are few tools to consider regularity questions in the RCD setting, we consider this problem on a *smooth* manifold $\mathcal{M}$ but with low-regularity metrics.

Rough metrics

Assume that $\mathcal{M}$ is a manifold (possibly noncompact).

Definition (Rough metric)

Let $\tilde{\mathrm{g}}$ be a $(2,0)$ symmetric tensor field with measurable coefficients and that for each $x \in \mathcal{M}$, there is some chart (U, ψ) near x and a constant $C \geq 1$ such that

$$C^{-1} \left|u\right|_{\psi^* \delta(y)} \leq \left|u\right|_{\tilde{\mathrm{g}}(y)} \leq C \left|u\right|_{\psi^* \delta(y)},$$

for almost-every $y \in U$ and where δ is the Euclidean metric in $\psi(U)$. Then we say that $\tilde{\mathrm{g}}$ is a rough metric, and such a chart (U, ψ) is said to satisfy the *local comparability condition*.

Induced measure

Define $\mu_{\tilde{g}}$ for a rough metric $\tilde{g}$ by writing

$$d\mu_{\tilde{g}}(x) = \sqrt{\det \tilde{g}(x)}\, d\mathscr{L}(x)$$

inside charts satisfying the local comparability condition and then patching them together via a partition of unity.

This measure is Borel-regular and finite on compact sets. It is unknown whether they are generally Radon. However, if $\mathcal{M}$ is compact, then it is.

Moreover, L^p theory exists (trivial) and ∇ on $\mathrm{C}^\infty \cap \mathrm{L}^2$ is a closable, densely-defined operator which gives Sobolev spaces $\mathrm{W}^{1,2}(\mathcal{M})$ and $\mathrm{W}^{1,2}_0(\mathcal{M})$.

Metric perturbations

Definition

We say that two rough metrics g and $\tilde{\mathrm{g}}$ are C-close if

$$C^{-1} |u|_{\tilde{\mathrm{g}}(x)} \leq |u|_{\mathrm{g}(x)} \leq C |u|_{\tilde{\mathrm{g}}(x)}$$

for almost-every $x \in \mathcal{M}$ where $C \geq 1$. Two such metrics are said to be C-close everywhere if this inequality holds for every $x \in \mathcal{M}$.

Note: on a compact manifold, there is always a C-close *smooth* metric g given a rough metric $\tilde{\mathrm{g}}$.

Proposition

Let g *and* $\tilde{\mathrm{g}}$ *be two rough metrics that are* C*-close. Then, there exists* $\mathrm{B} \in \mathbf{\Gamma}(\mathrm{T}^*\mathcal{M} \otimes \mathrm{T}\mathcal{M})$ *such that it is symmetric, almost-everywhere positive and invertible, and*

$$\tilde{\mathrm{g}}_x(\mathrm{B}(x)u, v) = \mathrm{g}_x(u, v)$$

for almost-every $x \in \mathcal{M}$*. Furthermore, for almost-every* $x \in \mathcal{M}$*,*

$$C^{-2} \left|u\right|_{\tilde{\mathrm{g}}(x)} \leq \left|\mathrm{B}(x)u\right|_{\tilde{\mathrm{g}}(x)} \leq C^2 \left|u\right|_{\tilde{\mathrm{g}}(x)},$$

and the same inequality with $\tilde{\mathrm{g}}$ *and* g *interchanged. If* $\tilde{\mathrm{g}} \in \mathrm{C}^k$ *and* $\mathrm{g} \in \mathrm{C}^l$ *(with* $k, l \geq 0$*), then the properties of* B *are valid for all* $x \in \mathcal{M}$ *and* $\mathrm{B} \in \mathrm{C}^{\min\{k,l\}}(\mathrm{T}^*\mathcal{M} \otimes \mathrm{T}\mathcal{M})$.

The measure $\mu_{\mathrm{g}}(x) = \theta(x)\ d\mu_{\tilde{\mathrm{g}}}(x)$, where $\theta(x) = \sqrt{\det B(x)}$.
Consequently,

(i) whenever $p \in [1, \infty)$, $\mathrm{L}^p(\mathcal{T}^{(r,s)}\mathcal{M}, \mathrm{g}) = \mathrm{L}^p(\mathcal{T}^{(r,s)}\mathcal{M}, \tilde{\mathrm{g}})$ with

$$C^{-\left(r+s+\frac{n}{2p}\right)} \|u\|_{p,\tilde{\mathrm{g}}} \leq \|u\|_{p,\mathrm{g}} \leq C^{r+s+\frac{n}{2p}} \|u\|_{p,\tilde{\mathrm{g}}},$$

(ii) for $p = \infty$, $\mathrm{L}^\infty(\mathcal{T}^{(r,s)}\mathcal{M}, \mathrm{g}) = \mathrm{L}^\infty(\mathcal{T}^{(r,s)}\mathcal{M}, \tilde{\mathrm{g}})$ with

$$C^{-(r+s)} \|u\|_{\infty,\tilde{\mathrm{g}}} \leq \|u\|_{\infty,\mathrm{g}} \leq C^{r+s} \|u\|_{\infty,\tilde{\mathrm{g}}},$$

(iii) the Sobolev spaces $\mathrm{W}^{1,p}(\mathcal{M},\mathrm{g}) = \mathrm{W}^{1,p}(\mathcal{M},\tilde{\mathrm{g}})$ and $\mathrm{W}^{1,p}_0(\mathcal{M},\mathrm{g}) = \mathrm{W}^{1,p}_0(\mathcal{M},\tilde{\mathrm{g}})$ with

$$C^{-\left(1+\frac{n}{2p}\right)} \|u\|_{\mathrm{W}^{1,p},\tilde{\mathrm{g}}} \leq \|u\|_{\mathrm{W}^{1,p},\mathrm{g}} \leq C^{1+\frac{n}{2p}} \|u\|_{\mathrm{W}^{1,p},\tilde{\mathrm{g}}},$$

(v) the divergence operators satisfy $\mathrm{div}_{D,\mathrm{g}} = \theta^{-1} \mathrm{div}_{D,\tilde{\mathrm{g}}} \theta \mathrm{B}$ and $\mathrm{div}_{N,\mathrm{g}} = \theta^{-1} \mathrm{div}_{N,\tilde{\mathrm{g}}} \theta \mathrm{B}$.

Note: Rough metrics are natural geometric invariances of the Kato square root problem. See [B2].

Assume now that $\mathcal{M}$ is compact. Let g be a rough metric and $\tilde{\mathrm{g}}$ a C-close smooth metric to g for some $C \geq 1$.

It makes sense to consider (GMC) in this context, provided we have the existence of a sufficiently good heat kernel.

Solving (GMC) is equivalent to solving

$$-\operatorname{div}_{\tilde{\mathrm{g}},y} \rho_t^{\mathrm{g}}(x,y) \mathrm{B} \theta \nabla \varphi_{t,x,v} = \theta \, \mathrm{d}_x(\rho_t^{\mathrm{g}}(x,y))(v), \tag{GMC'}$$

where $\mathrm{g}(\mathrm{B}u, v) = \tilde{\mathrm{g}}(u,v)$.

Concern: regularity of the metric

$$x \mapsto \mathrm{g}_t(x)(u,v) = \langle \rho_t^{\mathrm{g}}(x,\cdot) \nabla \varphi_{t,x,u}, \nabla \varphi_{t,x,v} \rangle_{\mathrm{L}^2(\mathrm{g})}.$$

Theorem (B., Lakzian, Munn ([BLM], 2015))

Let $\mathcal{M}$ be a smooth, compact manifold and g a rough metric. Let $\varnothing \neq \mathcal{N} \subset \mathcal{M}$ be an open set.

(i) *If the the heat kernel $(x, y) \mapsto \rho^{\mathrm{g}}_t(x, y) \in \mathrm{C}^{0,1}(\mathcal{M}^2)$ and improves to $(x, y) \mapsto \rho^{\mathrm{g}}_t(x, y) \in \mathrm{C}^k(\mathcal{N}^2)$ where $k \geq 2$. Then, for $t > 0$, g_t is a Riemannian metric on $\mathcal{N}$ of regularity $\mathrm{C}^{k-2,1}$.*

(ii) *If the heat kernel $(x, y) \mapsto \rho^{\mathrm{g}}_t(x, y) \in \mathrm{C}^1(\mathcal{M}^2)$ and $(x, y) \mapsto \rho^{\mathrm{g}}_t(x, y) \in \mathrm{C}^k(\mathcal{N}^2)$ where $k \geq 1$. Then, for $t > 0$, g_t is a Riemannian metric on $\mathcal{N}$ of regularity C^{k-1}.*

Standing question: What happens if we *only* assume that $(x, y) \mapsto \rho_t^{\mathrm{g}}(x, y) \in \mathrm{C}^1(\mathcal{N}^2)$ (i.e., no $\mathrm{C}^{0,1}$ or C^1 assumptions on global regularity). Is it true then that $x \mapsto \mathrm{g}_t(x) \in \mathrm{C}^0$?

Theorem (B. ([B], 2015))

Let $\mathcal{M}$ be a smooth, compact manifold, and $\varnothing \neq \mathcal{N} \subset \mathcal{M}$, an open set. Suppose that g is a rough metric and that $\rho_t^{\mathrm{g}} \in \mathrm{C}^1(\mathcal{N}^2)$. Then, g_t as defined by (GM) *exists on $\mathcal{N}$ and it is continuous.*

The equation (GMC) is a specific case of *pointwise* linear problems of the form:

$$\mathrm{L}_x u_x = \eta_x \tag{PE}$$

for suitable source data $\eta_x \in \mathrm{L}^2(\mathcal{M})$ and where $\mathrm{L}_x = -\operatorname{div} A_x \nabla$ is a family of divergence form operators.

Theorem (B. ([B], 2015))

Let $\mathcal{M}$ be a smooth manifold and g a smooth metric. At $x \in \mathcal{M}$ suppose that $x \mapsto A_x$ are real, symmetric, elliptic, bounded measurable coefficients that are L^∞-continuous at x, and that $x \mapsto \eta_x$ is L^2-continuous at x. If $x \mapsto u_x$ solves (PE) *at x with $\int_{\mathcal{M}} \eta_x \, d\mu_{\mathrm{g}} = 0$, then $x \mapsto u_x$ is L^2-continuous at x.*

Representation of solutions to the PDE

The equation (PE) can be further reduced to studying elliptic problems of the form

$$\mathrm{L}_A u = -\operatorname{div}_{\mathrm{g}} A \nabla u = f, \tag{E}$$

for suitable source data $f \in \mathrm{L}^2(\mathcal{M})$, where the coefficients A are symmetric, bounded, measurable and for which there exists a $\kappa > 0$ satisfying $\langle Au, u \rangle \geq \kappa \|u\|^2$.

By the operator theory of self-adjoint operators, we obtain that $\mathrm{L}^2(\mathcal{M}) = \mathcal{N}(\mathrm{L}_A) \oplus^{\perp} \overline{\mathcal{R}(\mathrm{L}_A)}$.

Moreover, $\mathcal{N}(\mathrm{L}_A) = \mathcal{N}(\nabla)$. Since $(\mathcal{M}, \mathrm{g})$ is smooth and compact, there is a Poincaré inequality, and since A are bounded below, $\mathcal{R} = \overline{\mathcal{R}(\mathrm{L}_A)} = \overline{\mathcal{R}(\sqrt{\mathrm{L}_A})}$, where

$$\mathcal{R} = \left\{ u \in \mathrm{L}^2(\mathcal{M}) : \int_{\mathcal{M}} u \; d\mu_{\mathrm{g}} = 0 \right\}.$$

Also, the embedding $E : \mathrm{W}^{1,2}(\mathcal{M}) \to \mathrm{L}^2(\mathcal{M})$ is compact, which implies that the spectrum of L_A is *discrete*. The Poincaré inequality implies a spectral gap between the zero and the first-nonzero eigenvalues.

Then, for $f \in \mathcal{R}$, $u = \mathrm{L}_A^{-1} f$ is a solution to (E) satisfying $\int_{\mathcal{M}} u \; d\mu_{\mathrm{g}} = 0$.

Back to the pointwise elliptic linear equation

Suppose that $\langle A_x u, u\rangle \geq \kappa_x \|u\|^2$, for $u \in \mathrm{L}^2(\mathrm{T}^*\mathcal{M})$.

Let $T_x = \sqrt{\mathrm{L}_x} = \sqrt{-\operatorname{div} A_x \nabla}$, and let $u_x, u_y \in \mathrm{L}^2(\mathcal{M})$ such that $\int_{\mathcal{M}} u_x \ d\mu_{\mathrm{g}} = \int_{\mathcal{M}} u_y \ d\mu_{\mathrm{g}} = 0$.

Then,

$$\|\mathrm{L}_x^{-1} u_x - \mathrm{L}_y^{-1} u_y\| = \|T_x^{-1} v_x - T_y^{-1} v_y\|$$

where $v_x = T_x^{-1} u_x$ and $v_y = T_y^{-1} u_y$. To prove L^2 continuity, it suffices to show that

$$\|T_x^{-1} v_x - T_y^{-1} v_y\| \lesssim \|A_x - A_y\|_\infty \|v_x\| + \|v_x - v_y\|.$$

Also,

$$\begin{aligned}\|T_x^{-1}v_x - T_y^{-1}v_y\| &\leq \|(T_x^{-1} - T_y^{-1})v_x\| + \|T_y^{-1}(v_x - v_y)\| \\ &\leq \|(T_x^{-1} - T_y^{-1})v_x\| + \|(T_y^{-1} - T_x^{-1})(v_x - v_y)\| \\ &\qquad + \|T_x^{-1}(v_x - v_y)\|.\end{aligned}$$

So, for $u \in \mathrm{L}^2(\mathcal{M})$ with $\int_{\mathcal{M}} u \, d\mu_{\mathrm{g}} = 0$,

$$\begin{aligned}\|T_x^{-1}u - T_y^{-1}u\| &= \|T_x^{-1}T_yT_y^{-1}u - T_x^{-1}T_xT_y^{-1}u\| \\ &= \|T_x^{-1}(T_y - T_x)T_y^{-1}u\| \lesssim \|(T_y - T_x)T_y^{-1}u\|\end{aligned}$$

Thus, it suffices to show

$$\|\sqrt{\mathrm{L}_x}u - \sqrt{\mathrm{L}_y}u\| \lesssim \|A_x - A_y\|_\infty \|\nabla u\|.$$

Such an estimate follows from *holomorphic dependency of the functional calculus* if we are able to prove a *homogeneous Kato square root estimate*.

Axelsson (Rosén)-Keith-McIntosh framework

(H1) The operator $\Gamma : \mathcal{D}(\Gamma) \subset \mathscr{H} \to \mathscr{H}$ is a closed, densely-defined and nilpotent operator, by which we mean $\mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$,

(H2) $B_1, B_2 \in \mathcal{L}(\mathscr{H})$ and there exist $\kappa_1, \kappa_2 > 0$ satisfying the accretivity conditions

$$\mathrm{Re}\,\langle B_1 u, u\rangle \geq \kappa_1 \|u\|^2 \text{ and } \mathrm{Re}\,\langle B_2 v, v\rangle \geq \kappa_2 \|v\|^2,$$

for $u \in \mathcal{R}(\Gamma^*)$ and $v \in \mathcal{R}(\Gamma)$, and

(H3) $B_1 B_2 \mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$ and $B_2 B_1 \mathcal{R}(\Gamma^*) \subset \mathcal{N}(\Gamma^*)$.

Let us now define $\Pi_B = \Gamma + B_1 \Gamma^* B_2$ with domain $\mathcal{D}(\Pi_B) = \mathcal{D}(\Gamma) \cap \mathcal{D}(B_1 \Gamma^* B_2)$.

Π_B is an ω-bisectorial operator with $\omega \in [0, \pi/2)$.

Quadratic estimates

To say that Π_B satisfies *quadratic estimates* means that

$$\int_0^\infty \|t\Pi_B(\mathrm{I} + t^2\Pi_B^2)^{-1}u\|^2 \ \frac{dt}{t} \simeq \|u\|^2, \tag{Q}$$

for all $u \in \overline{\mathcal{R}(\Pi_B)}$.

This implies that

$$\mathcal{D}(\sqrt{\Pi_B^2}) = \mathcal{D}(\Pi_B) = \mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma^* B_2)$$

$$\|\sqrt{\Pi_B^2}u\| \simeq \|\Pi_B u\| \simeq \|\Gamma u\| + \|\Gamma^* B_2 u\|$$

More importantly, for coefficients $A_1, A_2 \in \mathcal{L}(\mathscr{H})$ satisfying

(i) $\|A_i\|_\infty \leq \eta_i < \kappa_i$,
(ii) $A_1 A_2 \mathcal{R}(\Gamma), B_1 A_2 \mathcal{R}(\Gamma), A_1 B_2 \mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$, and
(iii) $A_2 A_1 \mathcal{R}(\Gamma^*), B_2 A_1 \mathcal{R}(\Gamma^*), A_2 B_1 \mathcal{R}(\Gamma^*) \subset \mathcal{N}(\Gamma^*)$,

we obtain that for an appropriately chosen $\mu < \pi/2$, and for all bounded holomorphic functions f in an open bisector containing the closed ω-bisector,

$$\|f(\Pi_B) - f(\Pi_{B+A})\| \lesssim (\|A_1\|_\infty + \|A_2\|_\infty)\|f\|_\infty. \tag{Hol}$$

This framework and connections to the Kato square root problem can be found in their paper [AKMc]. This is a first-order reformulation Kato square root problem resolved by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian in [AHLMcT].

The Kato square root problem on manifolds

Let $\mathscr{H} = \mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M})$, and set $S = (\mathrm{I}, \nabla)$

Set

$$\Gamma = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & S^* \\ 0 & 0 \end{pmatrix}.$$

$$B_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$$

for $a \in \mathrm{L}^\infty(\mathcal{M})$ and $A \in \mathrm{L}^\infty((\mathcal{M} \times \mathbb{C}) \oplus \mathrm{T}^*\mathcal{M})$.

Theorem (B., McIntosh ([BMc], 2012))

Let $(\mathcal{M}, \mathrm{g})$ *be a smooth, complete Riemannian manifold with* $|\mathrm{Ric}| \leq C$ *and* $\mathrm{inj}(M) \geq \kappa > 0$. *Suppose that the following ellipticity condition holds: there exist* $\kappa_1, \kappa_2 > 0$ *such that* $\mathrm{Re}\,\langle au, u\rangle \geq \kappa_1 \|u\|^2$ *and*

$$\mathrm{Re}(\langle A_{11}\nabla v, \nabla v\rangle + \langle A_{10} v, \nabla v\rangle + \langle A_{01}\nabla v, v\rangle + \langle A_{00} v, v\rangle) \geq \kappa_2 \|v\|^2_{\mathrm{W}^{1,2}}$$

for all $u \in \mathrm{L}^2(\mathcal{M})$ *and* $v \in \mathrm{W}^{1,2}(\mathcal{M})$. *Let* $D_A u = -a \,\mathrm{div}\, A_{11}\nabla u - a \,\mathrm{div}\, A_{10} u + a A_{01}\nabla u + a A_{00} u$. *Then, the quadratic estimates* (Q) *are satisfied,* $\mathcal{D}(\sqrt{D_A}) = \mathcal{D}(\nabla) = \mathrm{W}^{1,2}(\mathcal{M})$ *with* $\|\sqrt{D_A} u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{W}^{1,2}}$ *for all* $u \in \mathrm{W}^{1,2}(\mathcal{M})$, *and*

$$\|\sqrt{D_A} u - \sqrt{D_B u}\| \lesssim \|A - B\|_\infty \|u\|_{\mathrm{W}^{1,2}},$$

whenever b, B *are coefficients that satisfy accretivity assumptions with* $\eta_i < \kappa_i$

- Every smooth compact Riemannian manifold $(\mathcal{M}, \mathrm{g})$ satisfies the geometric assumptions: it is complete, $|\mathrm{Ric}| \leq C$, and there exists $\kappa > 0$ such that $\mathrm{inj}(\mathcal{M}, \mathrm{g}) \geq \kappa$.
- We want $\mathrm{L}_A = D_A$, but if we take A_{10}, A_{10} and A_{00} to be 0, we lose accretivity.
- The norm in the Lipschitz perturbation estimate is a $\mathrm{W}^{1,2}$ norm, but we need L^2 norms.
- The estimate of central importance here is the following coercivity estimate:

$$\|u\| \lesssim \|\Pi u\|,$$

for $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$. This is almost *trivial* for the inhomogeneous problem.

The homogeneous Kato square root problem on compact manifolds

Let $\mathscr{H} = \mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M})$.

Set

$$\Gamma = \begin{pmatrix} 0 & 0 \\ \nabla & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & -\mathrm{div} \\ 0 & 0 \end{pmatrix}.$$

$$B_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$$

for $a \in \mathrm{L}^\infty(\mathcal{M})$ and $A \in \mathrm{L}^\infty(\mathrm{T}^*\mathcal{M})$.

By self-adjointness for Π and, if the coefficients satisfy (H1)-(H3) by bi-sectoriality,

$$\mathscr{H} = \mathcal{N}(\Pi) \oplus^{\perp} \overline{\mathcal{R}(\Pi)} = \mathcal{N}(\Pi_B) \oplus \overline{\mathcal{R}(\Pi_B)}.$$

Thus, we have that $\mathrm{L}^2(\mathcal{M}) = \mathcal{N}(\nabla) \oplus^{\perp} \overline{\mathcal{R}(\mathrm{div})}$ and $\mathrm{L}^2(\mathrm{T}^*\mathcal{M}) = \mathcal{N}(\mathrm{div}) \oplus^{\perp} \overline{\mathcal{R}(\nabla)}$.

Moreover,

$$\overline{\mathcal{R}(\mathrm{div})} = \left\{ u \in \mathrm{L}^2(\mathcal{M}) : \int_{\mathcal{M}} u \; d\mu_{\mathrm{g}} = 0 \right\} = \mathcal{R}.$$

Now, let $u \in \mathcal{R}(\Pi) \cap \mathcal{D}(\Pi)$. So $u = (u_1, u_2)$, and

$$\|\Pi u\| = \|\nabla u_1\| + \|\operatorname{div} u_2\|.$$

Poincaré inequality then gives us that $\|\nabla u_1\| \geq C\|u_1\|$, since $u_1 \in \mathcal{R}(\operatorname{div})$.

Now, $u_2 = \nabla v_2$, for some $v_2 \in \mathcal{D}(\Pi)$. So,

$$\begin{aligned}\|\operatorname{div} u_2\| = \|\Delta v_2\| = \|\sqrt{\Delta}\sqrt{\Delta} v_2\| \geq C\|\sqrt{\Delta} v_2\| \\ = C\|\nabla v_2\| = C\|u_2\|.\end{aligned}$$

That is,

$$\|\Pi u\| \geq C\|u\|.$$

Theorem (B., ([B], 2015))

On a compact manifold $\mathcal{M}$ with a smooth metric g, the operator Π_B admits a bounded functional calculus. In particular, $\mathcal{D}(\sqrt{b \operatorname{div} B \nabla}) = \mathrm{W}^{1,2}(\mathcal{M})$ and $\|\sqrt{b \operatorname{div} B \nabla} u\| \simeq \|\nabla u\|$. Moreover, whenever $\|\tilde{b}\|_\infty < \eta_1$ and $\|\tilde{B}\|_\infty < \eta_2$, where $\eta_i < \kappa_i$, we have the following Lipschitz estimate

$$\|\sqrt{b \operatorname{div} B \nabla} u - \sqrt{(b + \tilde{b}) \operatorname{div}(B + \tilde{B}) \nabla} u\| \lesssim (\|\tilde{b}\|_\infty + \|\tilde{B}\|_\infty) \|\nabla u\|$$

whenever $u \in \mathrm{W}^{1,2}(\mathcal{M})$. The implicit constant depends on b, B and η_i.

Corollary

Fix $x \in \mathcal{M}$ and $u \in \mathrm{W}^{1,2}(\mathcal{M})$. If $\|A_x - A_y\| \leq \zeta < \kappa_x$, then for $u \in \mathrm{W}^{1,2}(\mathcal{M})$,

$$\|\sqrt{\mathrm{L}_x}u - \sqrt{\mathrm{L}_y}u\| \lesssim \|A_x - A_y\|_\infty \|\nabla u\|.$$

The implicit constant depends on ζ and A_x.

Corollary

Fix $x \in \mathcal{M}$ and suppose that $\|A_x - A_y\| \leq \zeta < \kappa_x$. Then,

$$\|\mathrm{L}_x^{-1}\eta_x - \mathrm{L}_y^{-1}\eta_y\| \lesssim \|A_x - A_y\|_\infty \|\eta_x\| + \|\eta_x - \eta_y\|,$$

whenever $\eta_x, \eta_y \in \mathrm{L}^2(\mathcal{M})$ satisfies $\int_\mathcal{M} \eta_x \, d\mu_\mathrm{g} = \int_\mathcal{M} \eta_y \, d\mu_\mathrm{g} = 0$. The implicit constant depends on ζ, κ_x, and A_x.

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